# MODELS IN MATHEMATICS TEACHING IN ITALY (1850-1950)

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**Abstract**

*Up to the present, the research on the collections of models in Italy has been limited to examining and cataloguing the collections existing at various universities, with a great deal of work being carried out mostly by Franco and Nicla Palladino,*[1] *but an in-depth study of their use in university and pre-university teaching does not yet exist.*

*In my paper I will deal with this problem, considering the period running from the mid-nineteenth century to the early decades of the twentieth century. In particular I will focus on the following points: the lack of interest in designing and making models for teaching at the university level at the turn of the twentieth century in Italy, and its causes; the one exception to this, i.e., Beltrami's cardboard model of the pseudospherical surface; models in pre-university schools in the 1800s; Corrado Segre and the use of models at the University of Turin; the role of models in the "laboratory school" (Vailati, Marcolongo, Montessori, Emma Castelnuovo); the reconstruction of university collections of models in the 1950s. Finally I will try to draw some conclusions from this first historical analysis of the question.*

## 1. The lack of interest in designing and making models for teaching at the university level at the turn of the twentieth century, and its causes

The use of models for research and teaching of mathematics began to spread in the second half of the nineteenth Century and saw mathematicians of the highest scientific calibre engaged in this effort. The most important initiatives developed in France (Paris), in the United Kingdom (London, Manchester) and especially in Germany (Munich, Darmstadt, Karlsruhe, Göttingen).[2] Although some first collections of models date from the early decades of the nineteenth Century, the mass production begun in the seventies mainly in Munich when Felix Klein came to teach at the local Technische Hochschule and began his collaboration with Alexander

[1] See the Bibliography.

[2] See for example N.& F. Palladino 2009.

Brill.[3] In the following years various exhibitions[4] and the publication of catalogues[5] favoured the spreading of the use of models in teaching at international level.

Italy remained marginal in the activity of conceiving and making geometric models, in spite of the many young mathematicians who went to Germany for post-graduate study.[6] Actually there was an attempt by Giuseppe Veronese (1854-1917) to set up a national laboratory for the production of models. In 1883 Veronese, who did post-graduate work in 1880-1881 in Leipzig where at the time Klein was teaching, in a report to the minister of education wrote:[7]

> *"I believe that for increasingly substantial progress of mathematical studies in Italy, an Atelier such as that in Munich would benefit us greatly, because our schools would be made independent in this respect too from foreigners and could procure their collections at less expense".*[8]

In fact he thought that

> *"Intuition in Geometry consists in our representing in our minds the figures in space, in such a way that our thoughts can go into them, alternately uniting them and separating them, and discovering the intimate nexus that permeates them. This is the intuition of space that must be developed in the minds of young people from their most tender years, and to that end it is useful to accompany each geometric proof, as far as possible, with drawings and models by which the young person can better comprehend and intuit the geometric properties of bodies without many mental efforts".*[9]

In spite of the support of important Italian mathematicians F. Brioschi, E. D'Ovidio, R. De Paolis, U. Dini and E. Bertini the initiative was unsuccessful.

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[3] See Rowe 2013.

[4] We mention the Exhibition in 1876 in London; the World Exhibition in 1885 in Anversa; the International Exhibition in 1893 in Monaco di Baviera; World's Columbian Exposition in 1893 in Chicago; the International Exhibition in Heidelberg during the Third International Congress of mathematicians in 1904; the Exhibition in 1914 in Edinburgh.

[5] See *Catalog der Modellsammlung des Mathematischen Instituts der kgl. Technischen Hochschule München, aufgestellt im Januar 1882 unter Leitung von Prof. A. Brill*, 1882 (see Fischer 1986, p. V; N.& F. Palladino 2009, pp. 51-52); *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente… herausgegeben von Walther Dyck, Professor an der technischen Hochschule München*, München, Wolf & Sohn, 1892, Nachtrag, in 1893; VII ed. 1903, VIII ed. 1911.

[6] Here are the best-known of these young mathematicians: A. Tonelli did post-graduate work in Göttingen (1877), A. Capelli in Berlin (1878), S. Pincherle in Berlin (1877-1878), G. Ricci Curbastro in Munich (1877-1878), L. Bianchi in Munich with Klein (1879-1880), G. Veronese in Leipzig with Klein, (1880–1881), E. Pascal in Göttingen with Klein (1888–1889), C. Segre visited Frankfurt, Göttingen, Leipzig, Nürnberg, Munich (summer 1891), G. Fano was in Göttingen with Klein (1893–1894), A. Viterbi in Göttingen, (1897–1898), F. Enriques in Göttingen (1903), etc. In general, 71% of students preferred to carry out their post-graduate studies in Germany, with only 18% opting for France; see A. Dröscher 1992, Die Auslandsstipendien der italienischen Regierung (1861-1894), Annali dell'Istituto storico italo-germanico in Trento, XVIII, pp. 545- 569.

[7] The following translation and all others in this paper are mine unless otherwise noted.

[8] F. Palladino 1999, p. 60. The report is reproduced in its entirety here.

[9] Ibid., pp. 59-60.

Starting in the 1880s, the principal Italian universities generally preferred to acquire models from abroad, mainly in Germany with reference to the catalogues of Brill and Schilling. The collections of some universities also included models from France (Muret Collection in Genoa) and England (George Cussons models of penetration in Naples). The first universities to acquire collections of models were those of Pisa, Rome, Turin, Pavia and Naples.

Initiatives for designing and producing models can be documented at the university of Naples; these however were generally limited to use within the university and there was no industrial production. In particular can be cited the 36 models (wood, bronze, horsehair) for the university teaching of descriptive geometry which Alfonso del Re (1859-1921) got his students to build between 1901 and 1906.[10] In this university the use of models in teaching practice was also favoured by Ernesto Pascal (1865-1940),[11] who specialised in Göttingen between 1888-1889 with Klein and, after some years spent in Pavia, arrived at Naples in 1907, where he remained until his death. Here, as Dean of the Faculty of Sciences he reorganised the teaching of mathematics, creating for each professorship a laboratory equipped with models and instruments. In fact be believed that models were useful in both teaching and in research; it is significant that Pascal mentions models several times in his book *Repertorio di matematiche superiori* (Milan, Hoepli, 1898, 1900),[12] translated in German under the title *Repertorium der höheren Mathematik* (Leipzig, 1900, 1902).

Very important was also the equipment of the Institute of rational mechanics directed from 1908 by Roberto Marcolongo (1862-1943), where geometric models in plaster or cardboard, often made by the students, were present alongside models and tools for the teaching of that discipline. From 1908 to 1934, in this Institute at least 134 degree theses were discussed, many of them accompanied by models. In his autobiography, Marcolongo writes:

> *"This ample and varied material is widely used, and is of the greatest aid in the lessons of rational and advanced mechanics.* There are no lessons in which, in one way or another, it is not shown to students*; indeed, some of the lessons are completely dedicated to illustrating theoretical results with experiments"*.[13]

Nevertheless, in Italy a systematic activity of designing and then constructing geometric models for university teaching at industrial level never flourished. It is significant, for example, that at the International Italian Exhibition, held in London in

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[10] See the booklet by Del Re with the title *Programma del corso e programma di esame per l'anno scolastico 1906-1907*, which in Appendix includes the list of geometric models built by the students of the school of descriptive geometry of the University of Naples from 1901 to 1906. Only one of these models has survived; see F. Palladino 1992, tav. 13.

[11] Pascal is well known for his integraphs for differential equations, see e.g. E. Pascal, I miei integrafi per equazioni differenziali*, Giornale di Matematiche*, (3), 51, 1913, pp. 369-375.

[12] See e.g. Pascal, *Repertorio di matematiche superiori*, op. cit. 1900, pp. 447, 469, 480, 750.

[13] Marcolongo 1935, pp. 30-31.

1888, the Education section was not awarded any prize because only publishing houses were present with books and other kinds of publication, and these had already been evaluated in their own section.[14] Many years later in 1904 at the exhibition of models organized during at the 3rd International Congress of Mathematicians (Heidelberg, 8-13 August 1904), Italy did not display anything, even though five Italian publishing houses were present.[15] The invitation to participate in that exhibition, extended to Italian mathematicians by the historian of mathematics Gino Loria in the pages of his *Bollettino di bibliografia e storia delle scienze matematiche*, [16] was ignored.

This situation seems to be connected to the fact that models were mainly used for educational purposes, at least according to the data known at present, and it was thus more convenient to purchase ready-made collections from abroad. Another explanation might lie in the very nature of scientific research: in the second half of the nineteenth century, in Italy three different approaches to geometric research were prevalent: the analytical approach, which was more theoretical and abstract (Ulisse Dini, Luigi Bianchi, etc.)[17]; the study of the foundations of geometry with an emphasis on logical rigour (Giuseppe Peano's School); and finally, the working method of the Italian School of algebraic geometry (Corrado Segre, Guido Castelnuovo, Federigo Enriques, etc.), whose members instead attributed great importance to intuition and visualisation. In spite of this, they did not use physical models in their research work, but preferred to employ the *Gedankenexperiment*. A famous passage by Castelnuovo describes the working method that he and Enriques used in their early years:

> *"We had constructed, in the abstract sense of course, a large number of models of surfaces in our space or in higher spaces; and we had distributed these models in two displays cases. One contained the regular surfaces ... At the end, the assiduous study of our models had led us to divine some properties which had to be true, with appropriate modifications, for the surfaces in both cases; we then put these properties to the test by constructing new models. If they stood up to the test, we looked for – ultimate phase – the logical justification"*.[18]

An analogous reference to a sort of virtual models seems to be in the following passage in a letter of Corrado Segre to Klein:

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[14] See *Esposizione Italiana di Londra 1888,* London: Waterlaw &Sons , 1888, pp. 231, 233.

[15] See *Modellaustellung, in Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, G. B. Teubner, Leipzig, 1905, pp. 731-736.

[16] See *Bollettino di bibliografia e storia delle scienze matematiche*, a. VII, 1904, p. 64. Loria periodically provided information on the construction of new models in his *Bollettino*.

[17] The theoretical results about surfaces with constant negative curvature by Dini and Bianchi are mentioned in the *Katalog ... von Walther Dyck*, p. 292.

[18] G. Castelnuovo 1928, p. 194.

> *What you tell me about the effect that synthetic reasonings of n-dimensional geometry has on you, does not surprise me*; it is only by living in Sn, (my emphasis), *by thinking about it always, that you become familiar with these arguments.*[19]

This attitude of the Italian geometers towards the geometric objects that they studied is also highlighted by Oscar Zariski's review of Beniamino Segre's treatise *The Non-singular Cubic Surfaces* (Oxford University Press, 1942). Zariski wrote:

> *"But the reader who is willing, so to speak*, to live for a while on a cubic surface (my emphasis) *and to read the book in the spirit in which it has been written will be greatly rewarded by the elegance and ingenuity of the author's treatment of the subject. It is based on the principle of continuity and the method of degeneration"*(Zariski 1943).

Another meaningful example that seems to show a mental use of models is the so-called 'Venetian lantern" used by Giuseppe Peano to prove that it is impossible to define the area of a curved surface as the limit of an area of an inscribed polyhedral surface when the longest of the diameters of the various faces tends to zero. The proof was given by Peano in an article dated 1890[20] (and contemporaneously by H. Schwarz). Peano's treatment is purely abstract and no figure appears.[21]

In university textbooks of the early twentieth century as well the representation of geometric models is also rather rare and serve solely didactic purposes. The only example found up to the present in which drawings of surfaces appear to recall actual models is the treatise *Lezioni di geometria intrinseca* (Naples, Presso l'Autore–Editore, 1896) by Ernesto Cesaro, in the part regarding surfaces of constant positive or negative curvature, and surfaces with a constant mean curvature (pp. 178, 181). In university textbooks of differential geometry and projective, descriptive or analytical geometry of the beginning of the century (L. Bianchi, U. Dini, F. Enriques, F. Severi, G. Castelnuovo, A. Terracini, G. Fubini and E. Čech) figures are either not introduced, or they are simple schematic representations of conic sections, quadric and pseudospherical surfaces.[22]

## 2. One exception: Beltrami's cardboard model of the pseudospherical surface

As illustrated above a systematic activity of creating and building geometric models for university teaching did not take roots among the Italian mathematicians.

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[19] C. Segre to F. Klein, Torino 11 May 1887, in Luciano and Roero 2012, p. 146.

[20] G. Peano 1890, Sulla definizione dell'area d'una superficie, *Atti della Reale Accademia dei Lincei: Rendiconti*, s. 4, 6, pp. 54-57.

[21] Instead in Schwarz's article (Sur une définition erronée de l'aire d'une surface courbe, *Gesammelte Mathematische Abhandlungen*, II, Berlin: Springer, pp. 309-311) one of the polyhedral surfaces of the sequence is represented.

[22] In Severi's *Vorlesungen über algebraische Geometrie*, Leipzig: Teubner, 1921 at p. 215 a model of a particular Riemann's surface is presented.

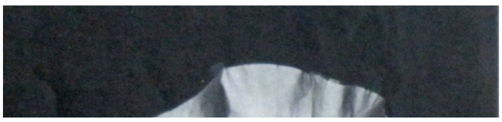
The only exception to this situation is represented by Eugenio Beltrami (1835-1900), who, as it is well known, in his *Saggio di interpretazione della geometria non euclidea* (1868) provided an interpretation of the lobačevskian plan by means of surfaces with constant negative curvature or pseudospherical, but was also interested in the material construction of these surfaces. From the correspondence with Jules Hoüel, we learn that he began to develop this interest starting in 1869:

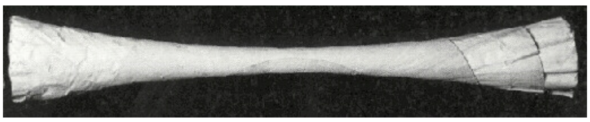
> *"In this period I have had a whimsical idea, which I shall tell you about...I wanted to try to construct materially the pseudospherical surface, on which the theorems of non-Euclidean geometry are fulfilled... If you consider the surface lying between two meridians, close enough together to allow it to be replaced, over a certain length, by a plane, you can, with little bits of paper cut into appropriate shapes reproduce the curved trapeziums whose true surface can be compounded".* [23]

With the aid of a pupil, Beltrami built several cardboard models, one of which is still preserved at the Institute of Mathematics of the University of Pavia. This is the copy[24] which he sent as a gift to his friend Luigi Cremona (1830-1903) on 25 April 1869 with a cover letter where he mentioned, amongst other things, a possible industrial-type production:

> *"I wouldn't want the model to remain at the Institute just to become food for the mice ... I would rather that it remain with you, in case one of your well-intentioned students with more patience than mine wished to try a more successful construction than the rough one I made. I also have a few other ideas for a more perfect execution to be made with other means and very different materials: but for this it is necessary to consult someone who is expert in industrial manipulations".*[25]

This cardboard model (Fig. 1), as Beltrami explains to Cremona, can be folded to represent the hyperbolic type of pseudospherical surface, or the parabolic type or simply the pseudosphere, but it cannot be folded to represent the elliptical type of pseudosphere without making a cut. For Beltrami it was not only a pleasurable pastime, but a tool for experimentally verifying the results obtained with theory and for visualising properties, as well as an artefact for discover new properties.





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[23] See the letter from Beltrami to Hoüel, Bologna 13 March 1869, in Boi, Giacardi, Tazzioli 1998, p. 80.

[24] This model is described by Beltrami in the letter to Hoüel, Bologna 22 April 1869, in Boi, Giacardi, Tazzioli 1998, p. 91.

[25] See the letter from Beltrami to Cremona, Bologna 25 April1869, in Boi, Giacardi, Tazzioli 1998, pp. 201-203.

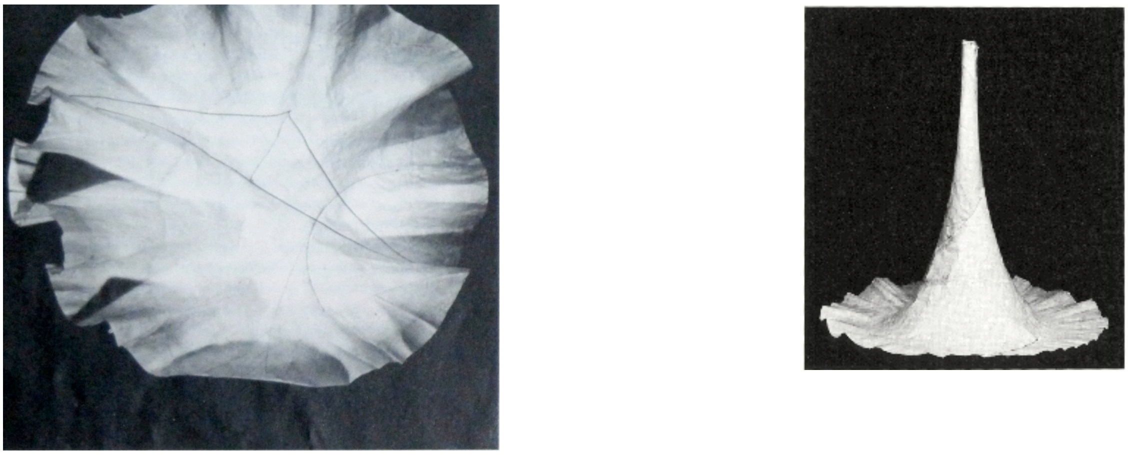

Fig. 1. a, left) Beltrami's model; b, top right) the model folded to represent the hyperbolic type of pseudospherical surface; c, bottom right) the model folded to represent the parabolic type.

For example, one new result that he obtained handling his cardboard model is the following (Fig. 2):

> *"Draw a straight line AB and at each of its points M draw the straight line MT which marks the direction of the parallel to AE, following Lobachevsky, with respect to the distance AM.* The envelope of these straight lines is the meridian of the pseudospherical surface. *The distance MN to the point of contact is constant"*.[26]

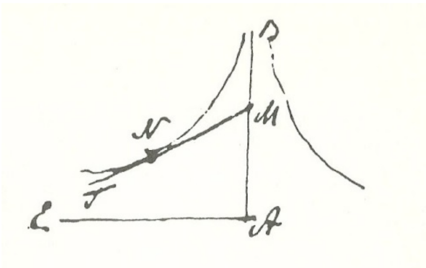

Fig. 2.

Beltrami published this result later in 1872.[27]

Another important result intuited by Beltrami through manipulating his cardboard model was the following that he communicated to Hoüel:

> *"You speak of empirical propositions that can be found by this means [the model], and you are perfectly correct, in fact here we have surfaces for which we possess no general equations. Here then is a proposition that I have begun to intuit: A pseudospherical surface can always be folded such that any whatsoever of its geodesic lines becomes a straight line. I give this to you only as an approximate result that is produced when, holding firm with both hands two points of the flexible surface, it is stretched as far as possible without tearing it. This result was even more striking to me*

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[26] Letter from Beltrami to Hoüel, Bologna 13 March 1869, in Boi, Giacardi, Tazzioli 1998, p. 82.

[27] E. Beltrami 1872, Teorema di geometria pseudosferica, *Giornale di matematiche* , 10, p. 53: Opere II, 392-393.

*because I supposed the opposite (I cannot say now on the basis of what arguments".*[28]

The last evidence of Beltrami's interest in the material construction of pseudospherical surfaces is the 1872 article *Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche*, aimed, as he said himself, "to prepare the geometrical elements – possibly easy and exact – of a material construction of the surface itself"[29].

After 1872 Beltrami abandoned this type of research and the correspondence with Hoüel shows this fact clearly, because afterwards no mentions to the construction of models appeared. His interests had progressively shifted toward issues regarding mechanics and mathematical physics.

When Beltrami told Cremona about his attempts to build a concrete model of pseudospherical surface, he wrote:

> *"The news that you are involved with the material construction of the pseudospherical surface pleases me greatly. These effective constructions are one of my dreams: but I don't know where to begin. I thus wait eagerly for you to communicate your results to me".* [30]

Cremona is the well-known mathematician, who developed the theory of birational transformations, but also the promoter of the return to the use of Euclid's *Elements* as a textbook in classical schools and therefore a supporter of a rigorous teaching of geometry. Notwithstanding, rigor was not the only aspect that characterized his vision of teaching: he also took in account *dynamic* aspects (based on the idea of transformation), *creative* aspects (which referred to the faculties of imagination, and the aesthetic sense), to which the *historical* aspect must be added.[31]

Despite this, as far as we know, it does not seem that Cremona built geometric models, but he was certainly interested in them. In addition to the letter to Beltrami this curiosity is attested by some letters to Thomas Hirst[32], where he inquired him about a model that Sylvester intended to build, or on models built by Plücker, or told about his visit in Munich during which Klein and Brill showed him the collection of geometric models. In the letters exchanged with Plücker Cremona asked him if there

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[28] Letter from Beltrami to Hoüel, Bologna 25 March 1869, in Boi, Giacardi, Tazzioli 1998, p. 86.

[29] E. Beltrami 1872a, Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche, *Giornale di Matematiche* 10, pp. 147-159, cit. at p.147; Opere II, 394-409.

[30] Letter by Cremona to Beltrami, Milano, 25 marzo del [1869]. The letter can be found on the website www.luigi-cremona.it. Beltrami kept a dense correspondence with Cremona: 1036 letters from 1864 to 1900.

[31] An example of this point of view is the article L. Cremona 1860, Considerazioni di storia della geometria in occasione di un libro di geometria elementare pubblicato a Firenze, *Il Politecnico*, 9, pp. 286-323 (*Opere matematiche*, Milano, Hoepli. v. I, 1914, pp. 176-207). See also Brigaglia 2006.

[32] See the letters from Cremona to Hirst: Bologna 18 January 1965, St. John Wood 13 February 1865, Bologna 24 March 1865, Milan 3 March 1868, 5 Oct 1876, in Nurzia 1999, pp. 72, 75, 77, 137, 175.

were photographs of the models built by Epkens for the Exhibition of Paris in 1867.[33] It is quite probable that the models and their use were also the topic of correspondence with Klein: in 1869 in a letter to Cremona, Klein underlined the importance of models:

> *"To me [this] section seems to be interesting, even apart from the theory of complexes, in so far as here a diverse family of surfaces is discussed in such a manner that the various shapes which arise are made evident. It has always seemed to me - and in this sense I understand Plücker's method when he had models constructed of the surfaces he dealt with there – that for geometrical problems not only are the theorems important that express relations between the objects to be treated but also important is the direct visualisation of these objects".*[34]

Furthermore, among the papers left by Cremona's daughter Itala Cozzolino to the Istituto Mazziniano in Genoa, there are beautiful cardboard models that Brill sent as gifts to Cremona, as it appears from the original envelope. They came from the first collection sold by Brill (fig. 3).[35]

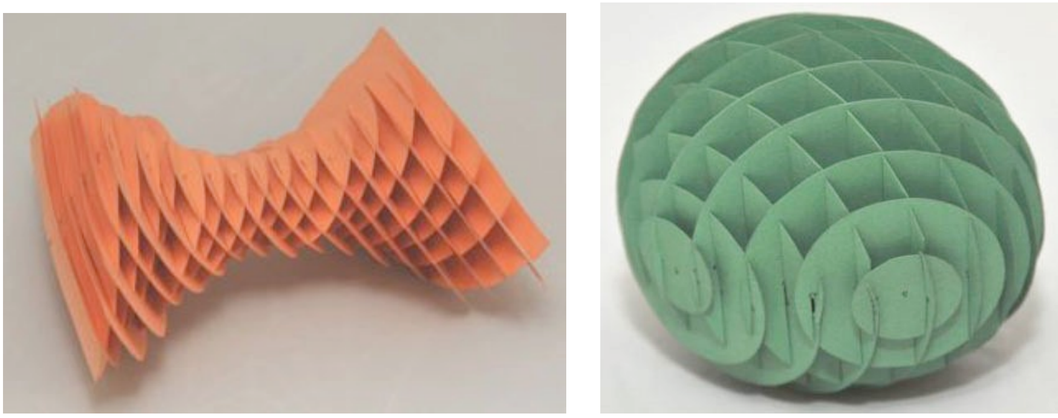

Fig. 3. Cardboard models of an ellipsoid and of a hyperboloid that belonged to Cremona

## 3. Models in the teaching of geometry at secondary schools at the end of the nineteenth century

Actually, in Italy the use of models had spread in mid-nineteenth century in the secondary schools and in the training of primary school teachers, in connection with the pedagogical movement promoted in Torino by the educators Ferrante Aporti, the founder of the first kindergartens (*asili aportiani*), Vincenzo Troya and Antonio

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[33] See the letters between Plücker and Cremona: Plücker to Cremona, Bonn 31 March 1867, Bonn 7 August 1867, Cremona to Plücker, Milan 5 April 1867, in Millán Gasca 1992, pp. 154-155, and Menghini 1994, p. 74. Other letters from Klein to Cremona are kept in the Istituto Mazziniano in Genoa, but they do not contain anything concerning the geometric models. I am grateful to Simonetta di Sieno for having provided me with these letters.

[34] Letter from Klein to Cremona, Göttingen, 6 July 1869, in Menghini 1994, p. 55. Other letters from ad to Klein can be found in: http://www.luigi-cremona.it/.

[35] See Rowe 2013.

Rayneri.[36] To go beyond the catechistic and repetitive methods in use up until that time, they maintained the importance of adopting, especially in primary teaching, the Socratic and intuitive methods, and the usefulness of basing teaching on children's experience and the manipulation of concrete objects.[37] In his book for primary teacher training, *Lezioni di nomenclatura geometrica* (Torino, Paravia, 1851, II ed. 1952) Rayneri explicitly discusses the use of geometric models (fig. 4), stating:

> *"It is to be desired that all students can observe them at their leisure, collocating them in diverse positions and comparing their various parts to each other and this is impossible if the objects are not submitted to a direct observation".* (p. XXXVI, II ed.)

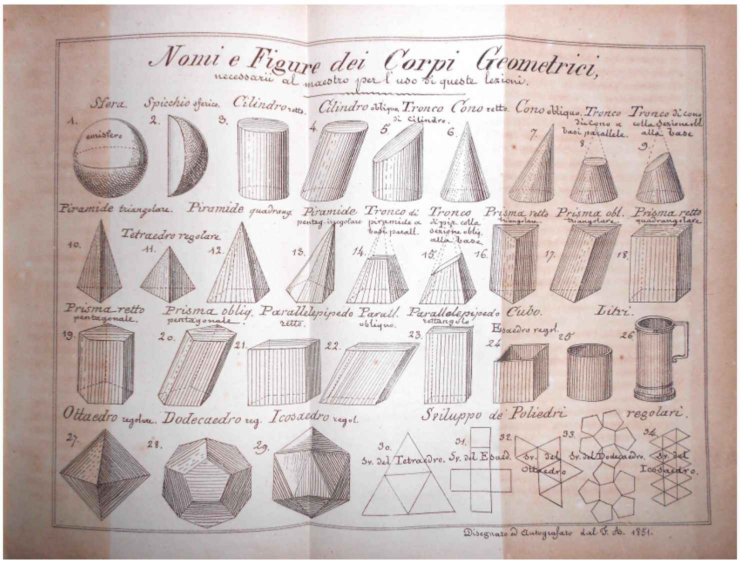


Fig. 4. Plate from Antonio Rayneri, *Lezioni di nomenclatura geometrica* (1851).

Under Rayneri's supervision, Giuseppe Crotti, professor of geometry and applied mechanics at the night school of the Workers Society in Torino, assembled two collections of geometric solids, one with 27 exemplars and the other of 35, sold in three sets (large, medium and small) by Paravia, a printer and bookseller in Torino, which at the time had begun to market educational aids of various kinds for schools.[38]

In the years that followed other publishing houses also began to publish catalogues of school materials, including collections of geometric models for classroom use. One such publisher was Giacomo Agnelli of Milan, whose catalogue for 1890-1891 presented a collection – available in six sets, including one of *stragrandissimi*

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[36] On the historical context see E. Luciano, C. Pizzarelli 2013, 'Educare è sinonimo di emancipare': le riviste della Società d'Istruzione e di Educazione, in F. Ferrara, L. Giacardi, M. Mosca (eds.) *Conferenze e seminari 2012-2013*, Associazione Subalpina Mathesis, Torino: Kim Williams Books, pp. 43-63.

[37] See C. Sideri 1999, *Ferrante Aporti, Sacerdote, italiano, educatore*. Milano: Franco Angeli, pp. 69-107.

[38] See *Giornale della Società d'Istruzione e di Educazione*, IV, 1852, pp. 109-110; see also the catolog by Paravia *Elenco di libri ed oggetti per le scuole normali-magistrali, elementari, tecniche, ginnasiali e liceali*, Torino, Milano: Paravia e Comp., 1862, p. 5.

(gigantic sizes) – of 24 demountable geometrical solids (fig. 5).[39] This free catalogue, of which 25,000 copies were printed, was sent to all schools.

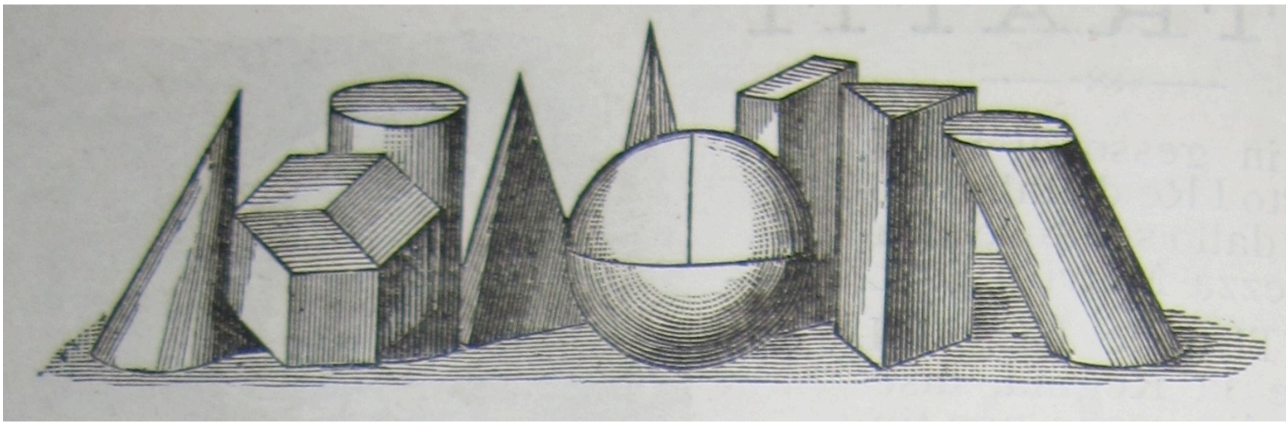

Fig. 5. The collection of demountable solids advertised in the Agnelli catalogue of 1890-1891.

Also particularly famous was the collection of crystallographical models conceived by Quintino Sella (1827-1884), one of the founders of mathematical crystallography, when he was teaching geometry applied to the arts in the 1850s at the Istituto Tecnico in Turin and later used in his classes in crystallography at the Scuola di Applicazione degli Ingegneri (the engineering school that later became the Politecnico di Torino).[40] Part of this beautiful collection is today conserved at the Istituto Tecnico Cavour in Vercelli (fig. 6).

Fig. 6. The collection of Quintino Sella's crystallographic models conserved today at the Istituto Tecnico Cavour in Vercelli.

The use of models for the teaching of geometry was prescribed by the school legislation: for example Article 152 of the 1853 Regulations for Schools for Teachers in primary and special schools of the Kingdom of Sardinia specified that such schools were to have geometric solids as part of their equipment,[41] and Art. 55 of the 1860 Regulations of the Scuola di Applicazione degli Ingegneri in Turin appear to indicate

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[39] *Libri ed articoli scolastici approvati per le Scuole del Regno,* Milano: Ditta Giacomo Agnelli, 1890-91, p. 84.

[40] See Q. Sella, *Lezioni elementari di Cristallografia* [*dettate alla Scuola d'Applicazione degli ingegneri di Torino nel 1861-62*], [Torino, Briola 1867] (lithograph).

[41] *Raccolta degli atti del governo di Sua Maestà il Re di Sardegna*, 1853, N. 1599, p. 1134.

that the “model-making workshop” of this School was given permission to build models for teaching for secondary schools in Piedmont.[42] In the Regulations (1867, 1892, 1895)[43] for *Scuole normali* (schools for elementary teachers training) of the Kingdom of Italy the use of models was recommended: in particular in the 1892 and 1895 Regulations, teachers were invited to “have the students construct the figures with cardboard, wire, etc., in order to better *derive a model from the drawings* done on the blackboard”.[44]

Mathematics programs of 1881, 1885 and 1890 for technical schools also invited teachers to use “large scale” models of geometric solids for teaching geometry.[45]

In the Paravia catalogues of school materials there appeared models of geometric solids first in wood or wire, and then also in cardboard,[46] and models of solids for teaching descriptive geometry appeared in collections that were increasingly varied and more beautiful up until the 1960s, often accompanied by the line “Collection recommended by the minister for public instruction”; for example, the catalogue of 1913-1914, in the section for “geometrical solids in painted wire, for the study of geometrical projections” carried an excerpt from the official bulletin of the Ministry for Education which underlined the benefits of the collection (fig. 7):

> *“The models further offer the advantage that they can be projected onto a surface: in fact, after the model has been placed in a certain position, it is sufficient to place a screen in front of it, and by properly illuminating the model itself, the shadow projected by the wires that make up its edges, will be drawn on the screeen, thus representing the projection of that figure on that given plane”*.[47]

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42 *Raccolta degli atti del governo di Sua Maestà il Re di Sardegna*, 1860, N. 4338, p. 1913, 1916. See also L. Sassi 1996, Rapporti istituzionali e legami culturali fra le scuole politecniche superiori e gli istituti tecnici e professionali secondari nel Piemonte post-unitario, *Le culture della tecnica,* 1, pp. 89-105, at p. 105.

43 *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1867, vol. VII p. 256; 1892, vol. IV p. 3622; 1895, vol. IV p. 4245.

44 *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1895, vol. IV p. 4280.

45 *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1881, 1885 and 1890, in *Documenti per la storia dell'insegnamento della matematica in Italia* (ed. by L. Giacardi and R. Scoth): http://www.mathesistorino.it/?page_id=564.

46 See, for example, *Catalogo del materiale scolastico e dei sussidi didattici per le scuole elementari,* Torino, Milano, Padova, Firenze, Napoli, Roma, Catania, Palermo: Paravia, 1937-1938, pp. 62-66.

47 See *Catalogo del materiale scolastico per gli asili infantili e per le scuole elementari*, Torino, Roma, Milano, Firenze, Napoli, Palermo: Paravia, 1913-1914, p. 57.

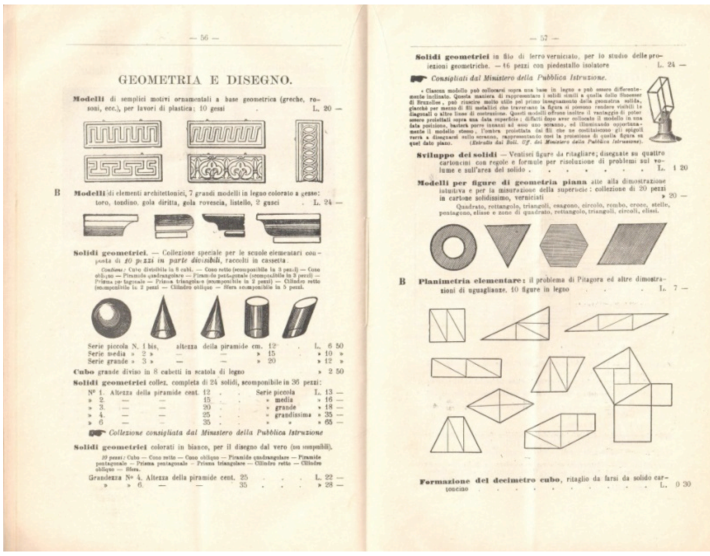


Fig. 7. Catalogue of school material for pre-schools and elementary schools (Paravia, 1913-1914, p. 57).


Fig. 8. Cover of the Vallardi catalogue of 1822-1823.

In the course of the twentieth century, other publishing houses in addition to Paravia also began to issue catalogues of school materials and distribute them by the thousands to Italian schools. One example was Antonio Vallardi of Milan, which in the 1960s also added to its catalogue models of geometric solids in plastic.[48] Another example is Arnoldo Mondadori of Verona, which began to produce this kind of

[48] See *Scuole elementari. Materiale didattico*, 8, Milano: Vallardi, 1961.

catalogue in 1927-1928 and also set up showrooms for this sort of materials in its offices in Verona, Rome, Milan and Turin.[49]

It is thus natural that the pedagogical congresses and the various national exhibitions often featured displays of models for use in secondary schools, as part of the section for education. For example, there were sections dedicated to education in the National Exhibition of 1858 in Turin, where two collections (of 200 pieces each) of Sella's wooden crystallographical models were displayed,[50] in the Pedagogical Congress held in Turin in 1869,[51] in the Italian Industrial Exhibition of 1884 in Turin[52] and in the National Exhibition of 1891 in Palermo, where solids made by the students of the Industrial School "Alessandro Volta" of Naples were exhibited.[53]

It is worth mentioning that in 1897 the publisher Hoepli in Milan printed the manual by Alfonso Rivelli *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta* addressed to secondary schools, which contains the explanation of how to build the various fundamental basic solids, but also regular star polyhedra – some of which were quite complicated – and the sphere. The book is enriched by various exercises which invite the student to apply what he has learned to the construction of new solids.

The representation of geometric models and their nets also appeared in several geometry textbooks for the lower level of secondary schools. For example, Giuseppe Veronese in his book *Nozioni di geometria intuitiva* (Verona: Drucker 1908) shows how to build the regular polyedra beginning with their nets-cut out of cardboard (pp. 94-98) because in this way, constructing and then handling models, the student is not obliged to follow passively the reasoning of the teacher, but can play an active part in the process of learning (p. VI). In the textbook by Angelo Pensa, *Elementi di geometria ad uso delle scuole secondarie inferiori* (Torino: Petrini, 1912), solids are represented with their nets, but the treatment is static and does not make any mention of the actual construction of models.

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49 See *Catalogo materiali didattici*, Verona: A. Mondadori, 1930, p. 9. Various of these catalogues are available in the Museo della Scuola e del Libro per l'Infanzia (MUSLI) in Torino, and the Museum has begun survey of didactic materials used in primary schools in Turin starting in the mid-1800s.

50 See *Relazione dei giurati e giudizio della R. Camera di Agricoltura e Commercio sulla Esposizione Nazionale di prodotti delle industrie, seguita nel 1858 in Torino*, Torino: Stamperia dell'Unione Tipografico-Editrice, 1860, p. 44

51 See *R. Museo industriale italiano. Illustrazioni delle collezioni. Didattica*, Torino, Napoli: dall'Unione Tipografico-Editrice, 1869.

52 La mostra didattica, Torino. *L'Esposizione italiana* 1884, N. 30 Torino, Milano: Roux e Favale

53 A. Rivelli, *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta*, Milan: Hoepli 1897, p. 10.

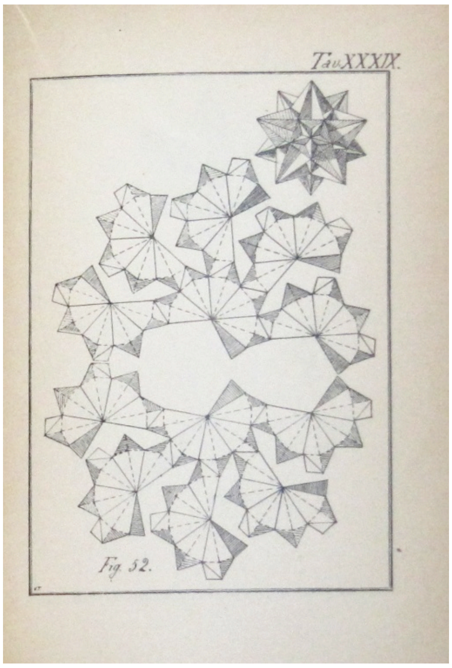


Fig. 9. Plate XXXIX from Alfonso Rivelli, *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta* (1897)

## 4. Corrado Segre and the use of models at the University of Turin

As already mentioned, the most important Italian universities bought collections of models from abroad. In Turin the first acquisitions date to 1880-1881, thanks to Enrico D'Ovidio (1843-1933), professor of higher geometry, rector of the University, director of the Teacher Training School and from 1883 director of the Mathematics Library.[54] The first to be purchased were surface models, in cardboard, plaster, wire, to be used for educational purposes. Documents from the archives of the "Giuseppe Peano" Library of Mathematics, tell us that, in January 1882, 47 plaster models by Brill of the first series I-VI, 7 cardboard models of quadric surfaces, 8 wireframe models by Björling at a total cost of 1,265.60 Italian lire, were purchased.[55]

In the years that followed, the models acquired by the Mathematics Library in Turin came mainly from the catalogues of Brill and of Schilling. Purchases are documented until 1919.[56] Among these also figure the five regular solids produced by Paravia. The documentation regarding the periodical restorations (by Fratelli Pallardi, Turin) of the models testify to the customary use of these in classes of advanced geometry[57] and in the mathematics lectures for the Teacher Training School.

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[54] E. D'Ovidio, Relazione delle cose più notevoli accadute durante l'anno scolastico 1880-81 nella R. Università di Torino, *Annuario, Università di Torino*, 1881-1882, pp. 3-7, cit. p. 7.

[55] See *Inventario… dal 1 aprile 1881 al 31 marzo 1883 in Dossier Inventari Consorzio*, Library of mathematics "Giuseppe Peano", Turin.

[56] See Giacardi 2003 and Ferrarese 2004.

[57] See, for example Segre's notebook *Superficie del 3° ordine e curve piane del 4° ordine* (1909-10): in the chapter concerning the cubic surfaces, Segre refers to models (p. 176), *Fondo Segre*, Library of mathematics "Giuseppe Peano", Turin, Quaderni 23.

In November 1907 Corrado Segre (1863-1924) took over the direction of the Library from D' Ovidio and held that position until his death. Segre – the founder of the Italian School of Algebraic geometry that numbers among its members outstanding mathematicians such as Guido Castelnuovo, Francesco Severi, Federigo Enriques, Gino Fano, Beppo Levi, Alessandro Terracini and Eugenio Togliatti – increased the collection of models, which he used in both his courses in higher geometry and in his lectures for the Teacher Training School:

> *"Corrado Segre gave classes on Tuesday, Thursday and Saturday mornings from 10 to 11, originally on the first floor in the lecture hall that occupied the place then used as the foyer of the Aula Magna, and later, I believe, in that lecture hall XVII of the second floor of the University building in Via Po, whose walls were lined with the glass cases with Brill's geometric models".*[58]

Fig. 10. Some of the models from the "Giuseppe Peano" Mathematics Libray of the University of Turin (Kummer surface, pseudosphere, Cayley surface, helicoid)

In fact, like Beltrami, Segre believed that the models can sometimes smooth the path to discovery, making it possible to "see certain properties that with deductive reasoning alone cannot be obtained".[59] This conviction derived directly from his way of conceiving scientific research, in which geometric intuition played a significant role, as he wrote to Klein:

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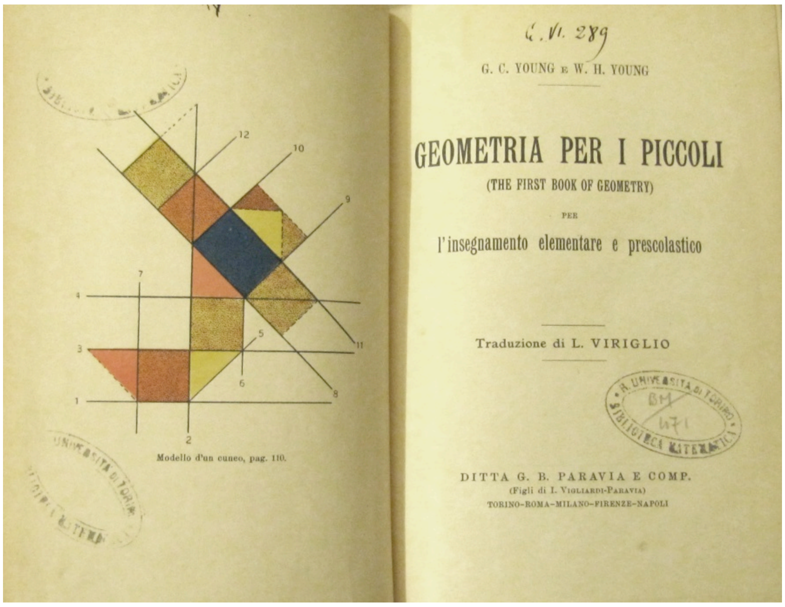
[58] Terracini 1968, p. 10.

[59] Segre, 1891, p. 54.

*“... the method that I like most, by my scientific inclinations, is the one mainly due to you: the geometric, or rather, synthetic, because it makes use of ingenious reasoning rather than lengthy calculations”*.[60]

This view also influenced his way to conceive the teaching of mathematics in secondary schools, as it clearly stands out from the manuscript notebook[61] of the lessons that for many years (from 1887/88 to 1890/91 and then again from 1907/08 to 1920/21) he taught at the Teacher Training School of the University of Turin. During his classes, in addition to dealing with elementary mathematics from an advanced standpoint, he also addressed didactic and methodological questions and, in particular, he believed that the first approach to mathematics should be experimental and intuitive, so that the student learns “not only to demonstrate truths already known, but to make discoveries as well, to solve the problems on his own”.[62] As far as geometry is concerned, Segre agreed with Giovanni Vailati’s point of view, according to which the teaching of this discipline should be experimental and operative and benefit from teaching aids such as squared paper, drawing, and geometric models. Segre’s notebook includes an annotated bibliography, where he lists not only papers by Vailati and Marcolongo, but also the little book by Rivelli mentioned




above and that by Karl Giebel, *Anfertigung mathematische Modelle für Schüler mittlerer Klassen* (Leipzig: Teubner 1915) on the construction of models for secondary schools teaching. With the aim of offering teachers a book inspired by this method, Segre invited one of his students, Luisa Virgilio, to translate Grace and William Young’s *A First Book of Geometry* (1905), in which the discovery of geometrical properties and theorems arises from the construction and manipulation of cardboard models. By folding paper, the authors guide the student towards simple “proofs” of some fundamental propositions. Then, they explain in detail how to build the regular polyhedra – or solids obtained from these by truncating them or placing two of them together – with appropriate cardboard “plane models” (with auxiliary faces), so that the student

[60] Letter from Segre to Klein, Torino 7 October 1884, in Luciano, Roero 2012, p. 134.

[61] See C. Segre, [Appunti relativi alle lezioni tenute per la Scuola di Magistero], Fondo Segre, Library of mathematics “Giuseppe Peano”, Turin, Quaderni 40. On this subject see Giacardi 2003a.

[62] Ivi, pp. 15, 16, 42.

> *“acquires not only manual dexterity, but complete familiarity with the truths which each model is meant to represent. Just because he can do this* by himself*, he is not taught, but learns, and he develops what may be called his geometrical sense”.*[63]

Fig. 11 - Grace and William Young, *Geometria per i piccoli* (translation by L. Viriglio), Torino: Paravia 1911.

The use of folding or cutting paper as part of teaching geometry was recommended by educators such as Pietro Pasquali (1847-1921), who believed in the educative value of manual work and maintained that the kind of teaching addressed to children should be fun and spontaneous. His publications included *Geometria intuitiva, ad uso delle scuole elementari superiori, tecniche, normali e industriali. Lezioni di ritaglio geometrico* (Parte I, Parma: Luigi Battei, 1901; Parte III, Milano: A. Vallardi, 1906).[64]

## 5. The role of models in the “laboratory of mathematics”

The book by Grace and William Young was inspired by a laboratory vision of mathematics teaching. The idea of a laboratory of mathematics[65] was introduced in the late nineteenth century by John Perry (1850-1920), a professor of mechanics and mathematics at the Royal College of Science in London, who proposed a new teaching method that he called ‘Practical Mathematics’, with emphasis on experiments, measurements, data gathering, drawing, graphic methods, and on the relationships between mathematics with physics and other sciences. With regard to geometry, Perry criticised the Euclidean method and suggested a teaching that was practical and experimental, heavily based on drawing, measuring and the use of squared paper. However, in his most famous book *Elementary Practical Mathematics* (London: Macmillan, 1913) no mention is made of geometric models.

Instead, in France, after the secondary school reform of 1902, Emile Borel (1871-1956) together with Jules Tannery (1848-1910) created the Laboratoire d’enseignement mathématique at the Ecole Normale Supérieure. This laboratory was aimed at training future teachers: here models in either wood or cardboard, wire and cork were conceived and built for teaching geometry and mechanics. The didactic uses of other instruments such as mechanical linkages, pantographs, inversors, calculating machines, and instruments for geodesy and land surveying were also

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[63] G.C. Young, W.H. Young, *A First Book of Geometry*, London: J. Dent, 1905, p. Viii.

[64] See for example G. Vacca 1930, Della piegatura della carta applicata alla geometria, *Periodico di Matematiche*, s. IV, X, pp. 43-50.

[65] For more on this subject see Giacardi 2011.

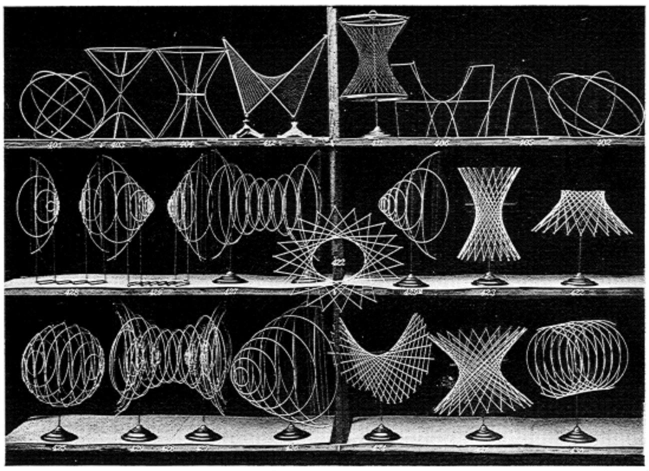
taught.[66] The stage of conceiving and then constructing models was a significant one, as Borel himself affirmed.[67]

Mentions of the use of models in secondary teaching can be found in the Meraner Lehrplan (1905),[68] prepared by the Unterrichtskommission der Gesellschaft deutscher Naturforscher und Ärzte (the Teaching Committee of the German Society of Natural Scientists and Physicians), adopting some of the basic points of Klein's reform movement. With regard to the teaching of geometry, the following aspects were emphasised: the strengthening of spatial intuition (p. 543); the use of straightedge and compass, drawing, measuring (p. 547); the consideration of geometrical configurations as dynamic objects (p. 548); making room for applications (p. 549); making use (*Benuzung*) of models; the coordination of planimetry and stereometry (p. 550).

The importance given by Felix Klein (1849-1925) to geometric models as an *Anschauungsmittel* (visual aid) in research and teaching of mathematics is well known and studied,[69] thus I only wish to point out that he frequently underlined the didactic function of models. For example in a lecture given in 1893 in Chicago he affirmed:

> *"The principal effort has been to reduce the difficulty of mathematical study by improving the seminary arrangements and equipments (sic)... Collections of mathematical models and courses in drawing are calculated to disarm, in part at least, the hostility directed against the excessive abstractness of the university instruction".*[70]

H. WIENERS UND P. TREUTLEINS SAMMLUNGEN MATHEMATISCHER MODELLE. Tafel III.



In his book *Anwendung der Differential-und Integralrechnung auf Geometrie* (Leipzig: Teubner 1907) Klein devoted a chapter, to the *Versinnlichung idealer Gebilde durch*

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[66] See A. Châtelet 1909, Le laboratoire d'enseignement mathématique de l'Ecole Normale Supérieure de Paris, *L'Enseignement mathématique*, 11, pp. 206-210.

[67] É. Borel 1904, Les exercices pratiques de mathématiques dans l'enseignement secondaire. *Revue générale des sciences pures et appliquées*, 15, 431-440, in Gispert 2002.

[68] See Bericht betreffend den Unterricht in der Mathematik an den neunklassigen höheren Lehranstalten, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 36, 1905, pp. 543-553. Also in F. Klein 1907, *Vorträge über den mathematischen Unterricht*, Teil 1, Leipzig: Teubner, pp. 208-219. See the English translation of the curriculum of Gymnasia in *The Mathematical Gazette*, 6. 95, 1911, pp. 179-181.

[69] See Rowe 2013.

[70] F. Klein 1894, *The Evanston Colloquium Lectures on Mathematics*, New York: Macmillan and Co, pp. 108-109.

*Zeichnungen und Modelle*, that is, the *Concretization of ideal objects through drawings and models* (p. 424).

Moreover, in his work on elementary mathematics from an advanced standpoint, in the volume on geometry, Klein presented various instruments, recommending that "the actual practical demonstration" not be neglected, when such instruments are considered in illustration of a theory.[71]

Klein was also involved in the reorganisation and modernisation of the *Modellkammer* in Göttingen for educational purposes, in particular to foster the *Raumanschauung* (spatial intuition),[72] and at the meeting of the International Commission on the Teaching of Mathematics (later ICMI) held in Brussels in 1910 he presented the geometric models from the collections of Brill and Schilling and illustrated their use in university teaching.[73] During that same meeting Peter Treutlein (1845-1912) presented a new series of models that were about to be released by the Teubner publishing house, and showed their use in secondary school.[74] In the catalogue – published two years later under the title *Verzeichnis mathematischer Modelle Sammlungen H. Wiener und P. Treutlein* (Leipzig: Teubner, 1912) – Treutlein devoted a section to models for teaching plane and solid geometry in secondary schools.

In Italy it was Giovanni Vailati (1863-1909), a mathematician, a philosopher and a member of the Peano School, who proposed an active approach to the teaching of mathematics, in the context of the work performed for the Royal Commission for the reform of secondary schools (1905-1909). He named this approach 'school as laboratory', not in the reductive sense of a laboratory for scientific experiments, but "as a place where the student is given the means to train himself, under the guidance and advice of the teacher, to experiment and resolve questions, to … test himself in the face of obstacles and difficulties aimed at provoking his perspicacity and cultivating his initiative".[75] In particular, according to Vailati, in the teaching of geometry the teacher should adopt an approach that is *sperimentale - operativo* (experimental and active ), using squared paper, drawing and geometric models, that is an approach directed "at developing not only the students' skills of observation but also those that come into play in the construction of the figures and in comparisons between them and their various parts, by means of measures, decompositions, movements, in short, by means of all those procedures that can lead to affirming and

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[71] F. Klein 1925, *Elementarmathematik vom höheren Standpunkte aus, II Geometrie*, Berlin: Springer, p. 16.

[72] For more on this, see Bartolini Bussi et al. 2010 and Schubring, 2010.

[73] See *L'Enseignement mathématique*, 12, 1910, pp. 391-392.

[74] Ibid., p. 388.

[75] G. Vailati 1906, Idee pedagogiche di H. G. Wells, in M. Quaranta (ed), *Giovanni Vailati, Scritti*, 3 vols., Forni, Bologna, 1987, III, pp. 291-295, at p. 292.

verifying their properties, which will later form the object of analysis and proof'.[76] So Vailati, like the Youngs, conceived a dynamic use of models in the teaching of geometry.

Various factors which cannot be further analysed here prevented the mathematics laboratory proposed by Vailati from becoming widespread in practice. First of all, the reform set forth by the Royal Commission was never carried through, second, Vailati never wrote a systematic exposition of his ideas and his premature death prevented any further developments. Third, laboratory method-inspired textbooks were never published in Italy, even though some authors paid attention to the geometrical constructions and to the experiments with folded or cut paper, sand, or small models in geometry. Fourth not all mathematicians in Italy shared Vailati's approach to teaching of mathematics and finally, his efforts would have been in any case nullified by the Gentile Reform of 1923, which made the humanities the cultural axis of national life in Italy, and especially of education (Giacardi 2011).

The only one who took up the idea of an effective laboratory-type teaching of mathematics at secondary schools was Marcolongo who, as seen above, had contributed to enrich the collection of models and mathematical instruments of the Rational mechanics Institute of the University of Naples. During the National Congress of the Mathesis (national association of mathematics teachers), held in Naples in October 1921, Marcolongo set up an exhibition of models and instruments, some of which were built by teachers or by students of secondary schools. Among others, wood or cardboard models of the principal elementary solids, the regular polyhedra, Dupin's decomposable solids, Brill's cardboard models of surfaces, Vuibert's anaglyphs, geometric figures for the stereoscope, and more were displayed.[77] Marcolongo also gave a lecture (Marcolongo 1922) on the use of educational and experimental materials in teaching practice, in which he illustrated his "ideal laboratory of mathematics". As he stated himself, he always had a passion for models and believed that the geometric experiments made by students through drawing and use of models "can not only promote geometric invention, and the discussion of problems, but also give a sense of confidence and mastery of the subject that is difficult to acquire by other means" (p. 7).

In a secondary school laboratory of mathematics, according to Marcolongo, there should be first of all, the models of solids of the elementary geometry and there should also be stereoscopes and stereoscopic figures useful to simulate three-dimensional vision and geometric anaglyphs by Henri Vuibert, that substantially create virtual geometric models. Moreover he was convinced that models had to be built by the students themselves in order to be really useful:

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[76] G. Vailati 1909, Sugli attuali programmi per l'insegnamento della matematica nelle scuole secondarie italiane. In *Atti del IV Congresso Internazionale dei Matematici, 6-11 aprile 1908*, Roma: Tip. Accademia dei Lincei, 482-487, pp. 484-485.

[77] The list of models and instruments can be found in Cardone 1996, pp. 148-149.

*"Paravia sells, for a modest price, a good collection of solids in wood, widely available but not as equally widely used, in all Italian schools. I have always had little sympathy with these collections; …they allow little or nothing of the inside to be seen, … Worse still, they are not built by students; they represent an experiment already made; instead the experiment must be made by the student***,** *naturally under the guidance of the professor. How much to be preferred are the models in cardboard, those made with thin wooden sticks connected to each other with a bit of wax and fine silk threads, built by the students!" (p. 8)… in knowing hands, the small model can be an inexhaustible source of simple experiments, easily verified, a starting point for observation and for the experimental discovery of new properties that the student will then attempt to prove by means that are logically rigorous... Make it so the professor comes to his classes armed with models, sheets of paper (even better if colored); that he has [the students] first observe, experiment, and then deduce and … everyone will see … a change of scene; the student will grow animated, will understand and (permit me the phrase) digest immediately"* (p. 9).

Marcolongo ended his lecture criticizing Italian mathematicians who too often disdained practical and experimental aspects of their discipline:

*"Our nation has had and still boasts of men of the highest merit both in the field of pure scientific speculation and in the field of pure experimentation; it has many, many fewer examples than other nations of that happy marriage of high scientific speculation with experimental ability that above all England and Germany can boast of"* (pp. 13-14).

He also observed that to reverse this trend it was necessary to start from the secondary schools.

## 6. The reconstruction of the university collections of models

The golden age for the construction of models in Germany came to an end with World War I, and in the 1930s production gradually ceased altogether, not only for reasons of marketing, but also because of the prevailing of a more abstract point of view in mathematical research.[78]

In Italy after World War II there was a revival of interest, due to the fact that in many Italian universities the collections had been destroyed during bombardments. In 1951 the Italian Mathematical Union[79], during its Fourth National Congress in Taormina, promoted the reconstruction of the collections of surface models in plaster or metal wire and Luigi Campedelli (1903-1978), mathematician from the University of Florence, was charged of this initiative. The Mathematics Institute of Pavia made

[78] The first model constructed after the war was that of the Peano surface, and even though there were new ones in preparation, their production was postponed due to unfavourable market conditions. In 1932 M. Schilling informed the Mathematical Institute in Göttingen that in the last few years no new exemplars had appeared; see Fischer 1986, p. X.

[79] Modelli per gli insegnamenti di Geometria e di Analisi, *Bollettino della Unione Matematica Italiana*, 3, VI, 1951, p. 366.

available its rich collection of models from Germany and some artisans in Florence looked after their reproduction.[80]

Various Italian universities acquired the collections, and these include the University of Turin which also retains the related documentation. Afterwards models appeared in some university publications such as those of Mario Villa[81] or of Campedelli[82] himself, who, among other things, treated the use of geometric models in the teaching at both secondary schools and university, in a large chapter of the collective book *Le matériel pour l'enseignement des mathématiques* (1958)[83] – the second promoted by la Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM) – which marked an important point in the history of mathematics education.

Later, gradually models came out from the university teaching and turned into beautiful display objects.

## 8. Models and a new way of teaching intuitive geometry in secondary school

One of the chapters of the book *Le matériel pour l'enseignement des mathématiques* was written by a young teacher Emma Castelnuovo (1913–), one of the daughters of Guido Castelnuovo, under the significant title *L'object et l'action dans l'enseignement de la géométrie intuitive* (The object and the action in the teaching of intuitive geometry*)*. In the 1940s she had been able to put into practice the ideas maintained by Vailati and Marcolongo, in fact she had introduced and experimented in his school, the Scuola Media Tasso (a lower secondary school) in Rome, a new way to teach intuitive geometry, a way that she called constructive to distinguish it from the descriptive one, generally in use up to that time. In Emma's approach the teaching material, drawings, models, are not considered as something static, through which the properties set out by the teacher should be verified, but they must be constructed and handled by the learner and thus become discovery tools. She declared:

> *"We want to emphasize that in any case, the material must be moveable: mobility is what in fact attracts the attention of the child, and that leads from concrete to abstract notions; because the subject of his attention is not*

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[80] See Primo elenco di modelli fatti costruire presso l'Università di Firenze a cura del prof. L. Campedelli, *Bollettino della Unione Matematica Italiana*, 3, VII, 1952, pp. 465-467; see also Modelli geometrici a cura del Prof. L. Campedelli, *Bollettino della Unione Matematica Italiana*, 3, VIII, 1953, p. 229, and Giacardi 2003.

[81] See M. Villa, *Lezioni di Geometria*, Padova: CEDAM, vol I 1965, Tables I-XII and vol. II Tables I-VIII with photographic reproductions of geometric models from the collection of the Istituto di Geometria "L. Cremona" of the University of Bologna, accompanied by very detailed captions.

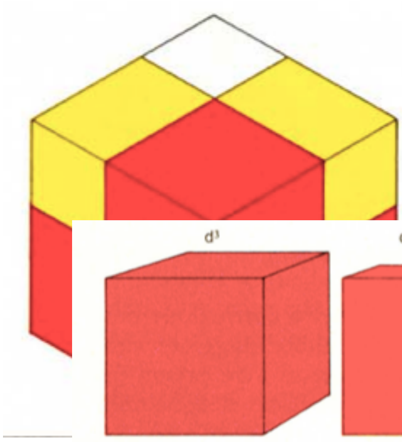
[82] See L. Campedelli, *Esercitazioni complementari di geometria*, Padova, CEDAM, 1955, Tables I-VIII; *Fantasia e logica nella matematica*, Milano: Feltrinelli 1966, pp. 49-50, Tables IV-VII.

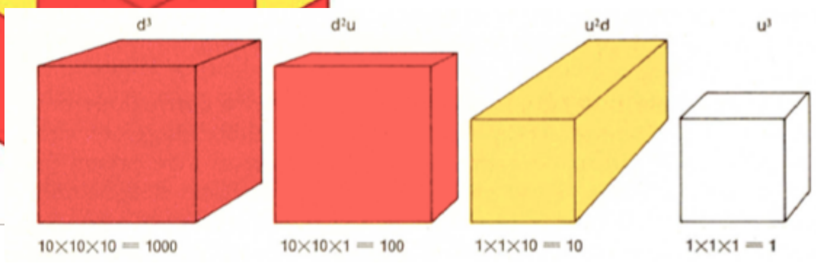
[83] L. Campedelli, *I modelli geometrici,* in *Il materiale per l'insegnamento della matematica*, Firenze: La Nuova Italia 1965, pp. 143-172.

*the material itself but rather the transformation of the material, an operation that, being independent from the material itself, is abstract”.*[84]

In these words of Emma it is possible to find an echo of the ideas of her uncle Federigo Enriques, who wrote:

*“… the construction of a geometric figure requires not only the attitude of passively seeing a model … but also the capacity to shape a possible model, on which are imposed, a priori, certain conditions: and this kind of constructive activity which orders the data of observations and past experience, is not pure imagination … but rather true logical activity”.*[85]






Emma also makes reference to Maria Montessori (1870-1952), the physician and educator who, at the beginning of the century, had introduced the use of special materials for teaching mathematics in primary schools (rods of different lengths, cubes, prism, etc.), and who in 1934 published the two books *Psico-aritmética* (Barcelona: Araluce, 1934) and *Psico-geométria* (Barcelona: Araluce, 1934), in which these materials were used in such a way as to permit, as Emma commented, not only a “grasp of various mathematical concepts by means of the senses, but also a grasp of operations”.[86] This is therefore an intuitive teaching based on a notion of intuition that is not only the “passive perception of an image or a material, but is also construction”.[87]

Fig. 10 Geometric visualisation of the cube of a binomial, from M. Montessori, *Psicoaritmetica*, Milano: Aldo Garzanti, 1971, pp. 334, 336.

[84] E. Castelnuovo, L’oggetto e l’azione nell’insegnamento della geometria intuitiva, Ibidem, pp. 41-65, at p. 58.

[85] F. Enriques, Insegnamento dinamico, Periodico di Matematiche 4, 1, pp. 6-16.

[86] E. Castelnuovo, L’insegnamento della matematica nella scuola preelementare e elementare, Scuola e Città, 3, 1957, pp. 93-98, at pp. 94-95.

[87] Ivi, p. 95.

Emma, however, also observed that "something is missing" for arriving to the intuition proper of a mathematician because with the Montessorian materials the mathematical experience is not practised on phenomena that vary with continuity. A peculiar characteristic of Emma's "intuitive geometry" is in fact the use of moveable models that can show the transformation from one figure to another, and thus stimulate mathematical intuition. Examples that can be cited are the parallelepiped constructed with small jointed rods,[88] and the cylinder built of elastic wires that can be transformed into a cone.[89] These are exercises that are possible today through the use of dynamic geometry software.

Emma was able to transform his classroom in a laboratory where everybody can perform free individual and collective creative work that trains both the mind and the character, anticipating some of the theoretical elaborations concerning the laboratory of mathematics that have characterized research in education in recent decades.[90]

## 9. A few conclusions

Further investigations into the archives of the Teacher Training Schools and the collections of the oldest technical institutes and classical high schools, a more extensive examination of unpublished scientific correspondence, manuscript lessons, textbooks, treatises, etc., could perhaps provide more accurate evidence of the use of models in mathematics teaching. However, the research carried out up to now concerning the period in consideration, allows us to draw some preliminary conclusions:

- In universities models were used as teaching aids as testified by the significant presence of collections in various Italian universities (F. & N. Palladino 2009);
- The creation of models did not greatly interest Italian mathematicians because of the nature of their research, and no adequate industry, comparable to that in Germany, had developed in Italy at the end of the 1800s;
- In primary school and in the lower-level secondary schools the presence of carton or wood models of elementary solids was widespread enough (Paravia's, Agnelli's, Vallardi's, Mondadori's collections, exhibitions, etc), even if their use seems often to be static, that is based only on the passive observation, also after the diffusion of the methods of the *école active* in Italy;
- In upper-level secondary schools it appears (textbooks, legislative measures, collections of models, etc.) that the use of models – for the study of elementary and descriptive geometry and of crystallography – was not infrequent, especially in technical and professional schools and institutes, while in *licei classici* (humanistic secondary schools) a teaching of a theoretical and abstract nature prevailed.

[88] E. Castelnuovo 1948, *Geometria intuitiva per le scuole medie inferiori*, Roma: Carabba, p. 160.

[89] E. Castelnuovo 1963, *Didattica della matematica*, Firenze: La Nuova Italia, pp. 105-109.

[90] For more on this subject and for the connections with the international research see L. Giacardi, R. Zan (eds.) 2013, Emma Castelnuovo. L'insegnamento come passione, *La Matematica nella Società e nella Cultura*, s.I, VI, pp. 1-193.

As the use of artefacts became accepted practice in mathematics teaching in Italy (Emma Castelnuovo) and in other countries (Gattegno, Châtelet, Piaget, etc.) in pre-university schools, models, in the broad sense of the term, acquired new importance. That importance has remained today, even though dynamic geometry software has partly replaced physical models with virtual models that are more ductile. However, the problem of whether the processes activated by the manipulation of software are the same as those activated by the manipulation of physical models is still open.[91]

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[91] On mathematical concrete and virtual manipulatives, see M. G. Bartolini Bussi, F. Martignone, Manipulatives in Mathematics Education, in Lerman, Steve (ed.), *Encyclopedia of Mathematics Education* in press.

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